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THE INFLUENCE OF TEMPERATURE DEPENDENT VISCOSITY ON LAMINAR BOUNDARY-LAYER STABILITY

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Abstract—The title problem is analyzed for small variations in viscosity by using a perturbation procedure. Consideration is restricted to incompressible, flat plate flow. The results show that appreciable stabilization may occur in water or other liquids, while air is slightly destabilized. Reasonable agreement with previous experimental results in air is obtained.

NOMENCLATURE

x, coordinate along plate [ft];
y, coordinate normal to plate [ft];
u, v, fluid velocity components in x and y directions respectively [ft/s];
T, fluid temperature [°K];
x^*, Reynolds number based on $x = \rho u_\infty x/\mu_w$;
f_1, f_2, functions shown in Fig. 1.

Greek symbols
ρ, fluid density [lbfs^2/ft^4];
μ, fluid viscosity [lbfs/ft^2];
α, thermal diffusivity [ft^2/s];
θ, dimensionless temperature, $= \dfrac{T - T_\infty}{T_w - T_\infty}$;
δ, δ_T, mechanical and thermal boundary-layer thickness respectively [ft];
η, η_T, dimensionless coordinates y/δ and y/δ_T respectively;
Δ, dimensionless ratio δ_T/δ;
λ, velocity profile shape factor;
ε, measure of viscosity variation, defined in equation (4);
δ^*, Reynolds number based on $\delta = \rho u_\infty \delta/\mu_w$.

Subscripts
∞, freestream conditions;
w, wall conditions;
0, zero order conditions (no heat transfer).

INTRODUCTION

It has long been recognized that the stability of a laminar boundary layer is markedly affected by heat transfer. In at least some cases, this phenomenon arises from changes in curvature of the velocity profile due to variation of fluid viscosity with temperature [1]. In this short note, an attempt is made to compute the change in the point of stability in a laminar, incompressible boundary layer on a flat plate held at constant temperature in a uniform flow. The von Kármán integral method is used to determine the variation of the velocity profile shape factor along the plate. Viscosity is assumed to change only slightly with temperature, and the resulting non-linear simultaneous differential equations are solved by a perturbation procedure. The resulting profile shape factor is then used to compute the instability criterion by adapting Schlichting's procedure for computing transition on airfoils [2] to the present problem.

ANALYSIS

For the sake of brevity, only the essential

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features of the computation are presented here, and the interested reader is referred to [6] for further details.

The appropriate boundary-layer equations for the problem stated in the introduction are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$
$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right), \quad (1)$$
$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \alpha\frac{\partial^2 \theta}{\partial y^2}.$$

Here x is the coordinate along the plate, and the fluid viscosity alone is assumed to vary with temperature.

We transform these equations by the usual von Kármán integral method, and following Pohlhausen assume velocity and temperature profiles of the form

$$u/u_\infty = 2\eta - 2\eta^3 + \eta^4 + \frac{\lambda}{6}(\eta - 3\eta^2 + 3\eta^3 - \eta^4),$$
$$\theta = 1 - 2\eta_T + 2\eta_T^3 - \eta_T^4. \quad (2)$$

The boundary-layer equations (1) become

$$\delta^* \frac{\mathrm{d}}{\mathrm{d}x^*}\left\{\delta^*\left[\frac{37}{315} - \frac{1}{945}\lambda - \frac{1}{9072}\lambda^2\right]\right\} = 2 + \frac{\lambda}{6},$$
$$\delta^* \Delta \frac{\mathrm{d}}{\mathrm{d}x^*}\{\delta^*\Delta[H_1(\Delta) + \lambda H_2(\Delta)]\} = \frac{2}{Pr_w}. \quad (3)$$

In these equations we have placed

$$\delta^* = \frac{\rho u_\infty \delta}{\mu_w}, \qquad x^* = \frac{\rho u_\infty x}{\mu_w},$$

where δ is the usual hypothetical boundary-layer thickness and Δ is δ_T/δ, the ratio of thermal to mechanical boundary-layer thicknesses. The functions $H_1(\Delta)$ and $H_2(\Delta)$ are polynomial expressions of Δ given by Dienemann [3], and the shape factor λ is determined by

$$\lambda = -\frac{12\varepsilon}{3\Delta + \varepsilon},$$

where

$$\varepsilon = \frac{1}{\mu_w}\left(\frac{\mathrm{d}\mu}{\mathrm{d}T}\right)_w (T_w - T_\infty). \quad (4)$$

Thus the quantity ε physically represents the percentage change in viscosity through the heated boundary layer, and may be algebraically positive or negative depending on the fluid considered. For slight variations in viscosity, ε can be made arbitrarily small. With fluids having moderate values of Prandtl number the quantity Δ is of order one, so that the parameter λ is of order ε. Moreover because of the choice of dimensionless quantities terms such as $\delta^*\,\mathrm{d}\delta^*/\mathrm{d}x^*$ are also of order one, while δ^* itself is of order 10^2. Hence for *small* variations in viscosity, we seek solutions to equations (3) by expanding δ^* and Δ in simple power series of the small parameter ε as follows;

$$\delta^* = \delta_0^* + \varepsilon\delta_1^* + \varepsilon^2\delta_2^* + \ldots,$$
$$\Delta = \Delta_0 + \varepsilon\Delta_1 + \varepsilon^2\Delta_2 + \ldots.$$

Substituting the above expressions for δ^* and Δ into equations (3) and collecting like powers of ε, we obtain a set of uncoupled linear, ordinary differential equations which were solved up to order ε^2. Since actual solutions for δ^* and Δ are only intermediate quantities in computing the stability criterion, they are not presented at this point.

Knowledge of the variation in λ along the plate from equation (4) and the solutions for δ^* now allows us to determine the point of instability. Schlichting and Ulrich [4] have computed the stability characteristics for a class of single parameter velocity profiles such as equations (2) by an Orr–Sommerfeld analysis. Reference [2] cites that for small values of λ, the critical Reynolds number (based on displacement thickness) is given by $Re_{cr} = 645 \exp(0{\cdot}6\,\lambda)$. By computing the Reynolds number based on displacement thickness along the plate and equating it to this critical value, we obtain a simple algebraic equation involving x^*, ε, and the Prandtl number Pr_w. Solving for x^*, the dimensionless point of instability may be expressed as

$$x^* = x_0^*[1 + \varepsilon f_1(Pr_w) + \varepsilon^2 f_2(Pr_w) + \ldots]. \quad (5)$$

The quantity x_0^* is the Reynolds number based on x at the point of instability with no heat transfer.

RESULTS AND DISCUSSION

The functions f_1 and f_2 have been computed for a range of Prandtl numbers and are shown in Fig. 1. These functions can be approximately expressed as

$$f_1 \simeq 4{\cdot}65\, Pr_w^{1/3},$$

$$f_2 \simeq 15{\cdot}5\, Pr_w^{2/3},$$

so that the ratio of Reynolds number at the point of instability to the same quantity with

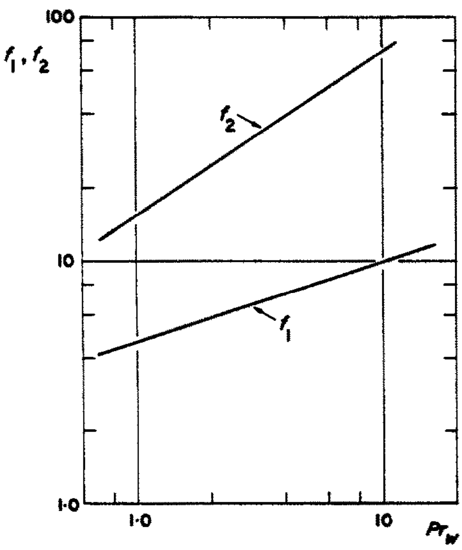

FIG. 1. The functions f_1, f_2 vs. Prandtl number.

no heat transfer present may finally be given as

$$\frac{x^*}{x_0^*} = 1 + 4{\cdot}65\, Pr_w^{1/3}\varepsilon + 15{\cdot}5\, Pr_w^{2/3}\varepsilon^2 + \ldots. \quad (6)$$

Equation (6) has been plotted in Fig. 2 for air and water at 1 atm and 20°C. It can be seen that

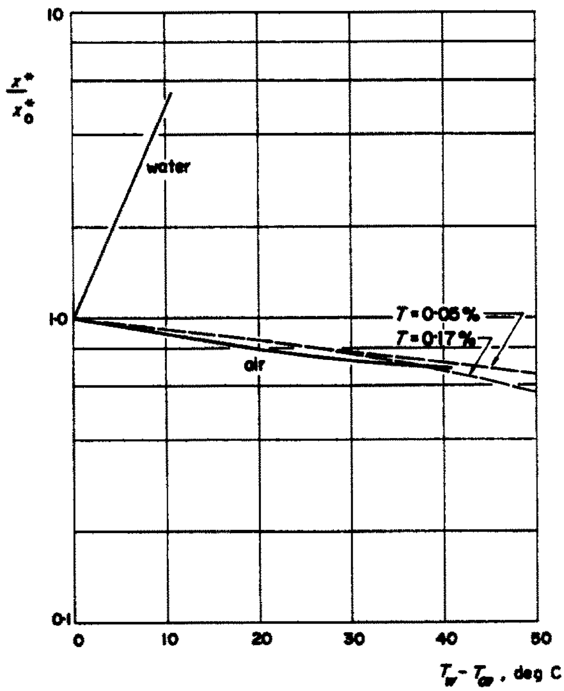

FIG. 2. Ratio of Reynolds number at the point of instability to the same quantity with no heating, vs. temperature differential (for air and water at 1 atm and 20°C). Dashed lines from [5]; T is the freestream turbulence level.

very strong stabilization should occur in water with as little as 10 degC temperature differential, while air is destabilized only with much larger differentials.

As far as the author is aware, the only direct experimental measurements of transition as affected by heat transfer (in subsonic flow) are those of Liepmann and Fila [5]. Their results have been replotted in Fig. 2, along with results predicted by equation (6). In comparing the results for air, it should be noted that equation (6) refers to the point of instability, whereas the experimental results are for the point of transition, so that a direct comparison is not valid. Nonetheless equation (6) predicts the correct trend of result, and the curve shown for water in Fig. 2 presumably may be used with reasonable confidence.

Finally it must be realized that actual transition is more complicated a phenomenon than Schlichting's procedure admits, and the present results should only be regarded as estimates. Reference [2] contains a critical review of Schlichting's method.

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Résumé—Le problème énoncé dans le titre est analysé pour de petites variations de la viscosité en employant une méthode de perturbation.

On s'est restreint à l'écoulement incompressible le long d'une plaque plane. Les résultats montrent qu'une stabilisation appréciable peut se produire dans l'eau ou d'autres liquides, tandis que l'on a une légère déstabilisation dans l'air. Un accord raisonnable est obtenu avec des résultats expérimentaux antérieurs.

Zusammenfassung—Das in der Überschrift genannte Problem wird für kleine Änderungen der Zähigkeit mit Hilfe eines Strömungsverfahrens analysiert. Die Betrachtungen sind auf die inkompressible Strömung entland einer ebenen Platte beschränkt. Die Ergebnisse zeigen, dass in Wasser und anderen Flüssigkeiten eine merkliche Stabilisierung eintreten kann, während Luft geringfügig instabiler wird. Zufriedenstellende Übereinstimmung mit neueren Versuchsergebnissen an Luft wird erhalten.

Аннотация—Методом возмущений рассматривается задача о влиянии вязкости, зависимой от температуры, на устойчивость ламинарного пограничного слоя при небольщих изменениях вязкости.

Рассмотрение ограничено обтеканием плоской пластины несжимаемым потоком жидкости. Результаты показывают, что стабилизация имеет место для воды и других жидкостей, тогда как поток воздуха несколько неустойчив.

Получено хорошее соответствие с уже имеющимися данными для воздуха.